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Periodic Quadratic Spline Interpolation*

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1. INTRODUCTION

Let $\Delta = \{x_i\}_{i=0}^{N}$ be a *partition* of $[a, b]$, $a = x_0 < \cdots < x_N = b$. The *length* of the interval $[x_i, x_{i+1}]$ is $h_i = x_{i+1} - x_i$ $(i = 0,..., N-1)$, the *mesh size* of the partition is $\|\Delta\| = \max_i h_i$ and the *mesh ratio* of the partition is $\gamma(\Delta) = \|\Delta\|/\min_i h_i$. A partition Δ is *uniform* if its mesh ratio $\gamma(\Delta) = 1$. A family of partitions is *regular* if there exists a strictly positive constant γ such that $\gamma(\Delta) \geqslant \gamma$ for each partition Δ in the family.

A *quadratic spline* s is a function $s \in C^1[a, b]$ such that s restricted to $[x_i, x_{i+1}]$ is a polynomial of degree $\leqslant 2$. It is a *periodic quadratic spline* if $s^{(1)}(a) = s^{(1)}(b)$ (the condition $s(a) = s(b)$ is not used here).

Throughout this paper we will use the following notations. If g: $[a, b] \to R$ is a given function, we will write $g_i = g(x_i)$, $x_{i+1/2} = (x_i + x_{i+1})/2$ and $g_{i+1/2} = g(x_{i+1/2})$. For a positive integer N we will note Z_N the set $\{0, 1,..., N-1\}$ and Z_N^e (resp. Z_N^0) the set of even (resp. odd) numbers in Z_N.

In this paper we define a periodic quadratic spline from its nodal values $s_i (i = 0,..., N)$. In Section 2, we recall an existence and uniqueness result and we give an explicit representation for the moments $s_i^{(1)}$ $(i = 0,..., N)$. In Section 3, if s is the periodic quadratic spline interpolant of $f \in C[a, b]$, we obtain error bounds of the form $\|f^{(l)} - s^{(l)}\|_\infty \simeq O(\|\Delta\|^{k+1-l})$ $(0 \leqslant l \leqslant k+1$, $0 \leqslant k \leqslant 2)$ which are valid only when the partition Δ is uniform.

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TABLE I

Summary of the Convergence Results: $\|f - s\|_q \simeq O(\|\Delta\|^m)$

$m = 1$	$f \in C[a, b], f^{(1)} \in BV[a, b]$		Theorem 4
$m = 2$	(i) $f \in AC_p^{2,q}[a, b], f^{(2)} \in BV[a, b]$,	uniform Δ	Theorem 5 $(k = 1)$
	(ii) $f \in AC_p^{3,1}[a, b]$,	regular Δ	Theorem 7
$m = 3$	$f \in AC_p^{3,q}[a, b], f^{(3)} \in BV[a, b]$,	uniform Δ	Theorem 5 $(k = 2)$

Table I gives a summary of our main results. In this table, and throughout this paper, we use the following notations:

$$AC^{k+1,q}[a, b] = \left\{ f \in C^k[a, b] \;\middle|\; \begin{array}{l} (a) f^{(k+1)} \in L^q[a, b] \\ (b) f^{(k)}(s) = f^{(k)}(r) + \int_r^s f^{(k+1)}(\xi)\, d\xi, \forall r, s \in [a, b] \end{array} \right\},$$

where $1 \leqslant q \leqslant \infty$ and $k \geqslant 0$, and

$$BV[a, b] = \{f\colon [a, b] \to R \mid \operatorname{Var}(f) < \infty\},$$

where $\operatorname{Var}(f)$ is the total variation of f on $[a, b]$. Moreover,

$$f \in AC_p^{k+1,q}[a, b] \qquad \text{if} \quad f \in AC^{k+1,q}[a, b] \text{ and } f^{(1)}(a) = f^{(1)}(b).$$

These results are extensions, to the periodic case, of those obtained by J. W. Daniel [2] and C. de Boor [1]. Finally, other quadratic spline interpolation approaches have been proposed before, for instance, see Kammerer *et al.* [5], M. J. Marsden [7], S. Demko [3], E. Neuman [9] and Sharma and Tzimbalario [10].

2. Existence of Periodic Quadratic Splines

As previously defined, on each interval $[x_i, x_{i+1}]$ a periodic quadratic spline can be written

$$s(x) = s_i + (x - x_i)\, s_i^{(1)} + \frac{(x - x_i)^2}{2h_i}\,(s_{i+1}^{(1)} - s_i^{(1)}).$$

Consequently

$$s_i^{(1)} + s_{i+1}^{(1)} = 2\,\frac{s_{i+1} - s_i}{h_i} \qquad (i = 0, \dots, N - 1), \tag{1}$$

and this leads us to the following result (see also Meinardus and Taylor [8] and Krinzesza [6]).

THEOREM 1. *Let $\Delta = \{x_i\}_{i=0}^N$ be a partition of $[a,b]$. A periodic quadratic spline is uniquely determined by its nodal values $\{s_i\}_{i=0}^N$ if and only if N is odd. In this case*

$$\begin{bmatrix} s_0^{(1)} \\ s_1^{(1)} \\ s_2^{(1)} \\ \vdots \\ s_{N-1}^{(1)} \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 & \cdots & -1 & 1 \\ 1 & 1 & -1 & \cdots & 1 & -1 \\ -1 & 1 & 1 & \cdots & -1 & 1 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ -1 & 1 & -1 & \cdots & 1 & 1 \end{bmatrix} \begin{bmatrix} (s_1 - s_0)/h_0 \\ (s_2 - s_1)/h_1 \\ (s_3 - s_2)/h_2 \\ \vdots \\ (s_N - s_{N-1})/h_{N-1} \end{bmatrix}. \tag{2}$$

If N is even, the spline does not exist or is not uniquely determined.

Proof. If we use the assumption of periodicity $s_0^{(1)} = s_N^{(1)}$, the matrix form of (1) is $As^{(1)} = b$, where

$$A = \begin{bmatrix} 1 & 1 & & & 0 \\ & 1 & 1 & & \\ & & \cdots & & \\ & 0 & & 1 & 1 \\ 1 & & & & 1 \end{bmatrix}, \quad s^{(1)} = \begin{bmatrix} s_0^{(1)} \\ \vdots \\ s_{N-1}^{(1)} \end{bmatrix} \quad \text{and} \quad b = 2 \begin{bmatrix} (s_1 - s_0)/h_0 \\ \vdots \\ (s_N - s_{N-1})/h_{N-1} \end{bmatrix}.$$

Then $\det A = 1 + (-1)^{N+1}$ and the result follows. Q.E.D.

3. Derivation of Error Bounds

Given a function $f: [a,b] \to R$ and a partition $\Delta = \{x_i\}_{i=0}^N$, *N odd*, of the interval $[a,b]$, we consider the periodic quadratic spline interpolant s of f such that $s(x_i) = f(x_i)$. By definition, the *remainder function* or *error* is $e(x) = f(x) - s(x)$. In this section, we derive uniform bounds for the remainder function. Thus we extend the results of J. W. Daniel [2] and C. de Boor [1] to the periodic quadratic spline interpolation.

3.1. *Preliminary Results*

The study of the remainder function e rests on the behaviour of $e_i^{(1)}$ $(i = 0, \ldots, N)$.

PROPOSITION 2. *Let $k = 0$, 1 or 2 and $f \in AC^{k+1,\infty}[a,b]$. If there exists a constant C_k and a real number α such that*

$$\max\{|e_i^{(1)}|, |e_{i+1}^{(1)}|\} \leqslant C_k h_i^\alpha \tag{3}$$

for all $i \in Z_N$, *then there exist constants* C_{kl} *which depend only on* C_k *and* $\|f^{(k+1)}\|_\infty$, *such that for almost all* $x \in [x_i, x_{i+1}]$

$$|e^{(l)}(x)| \leqslant C_{kl}[h_i^{\alpha+1-l} + h_i^{k+1-l}]$$

for all $l = 0, ..., k+1$ *and* $i \in Z_N$ (*when* $k = 2$ *and* $l = 3$ *we rather have* $\|e^{(3)}\|_\infty = \|f^{(3)}\|_\infty$).

Proof. A direct adaptation of Stoer and Bulirsch's [11] Theorem 2.4.3.3 (see Dubeau and Savoie [4, Proposition 3.1]). Q.E.D.

We try now to obtain bounds of the form (3). A first step in this way is

PROPOSITION 3. *Let* $k = 0$, 1 *or* 2 *and* $f \in AC^{k+1,\infty}[a,b] \cap C^1[a,b]$. *Then there exists a constant* C_k, *independent of the partition, such that*

$$|e_i^{(1)} + e_{i+1}^{(1)}| \leqslant C_k h_i^k \|f^{(k+1)}\|_\infty \tag{4}$$

for all $i \in Z_N$. *Moreover,* $C_0 = 4$, $C_1 = 1/2$ *and* $C_2 = 1/6$.

Proof. From (1) we always have

$$e_i^{(1)} + e_{i+1}^{(1)} = f_i^{(1)} + f_{i+1}^{(1)} - \frac{2}{h_i}\int_{x_i}^{x_{i+1}} f^{(1)}(\xi)\, d\xi$$

and $C_0 = 4$. If $k = 1$, through integration by parts, we obtain

$$e_i^{(1)} + e_{i+1}^{(1)} = \frac{2}{h_i}\int_{x_i}^{x_{i+1}} (\xi - x_{i+1/2}) f^{(2)}(\xi)\, d\xi$$

and $C_1 = 1/2$. If $k = 2$, through integration by parts agains, we obtain

$$e_i^{(1)} + e_{i+1}^{(1)} = \frac{h_i}{4}\int_{x_i}^{x_{i+1}} f^{(3)}(\xi)\, d\xi - \frac{1}{h_i}\int_{x_i}^{x_{i+1}} (\xi - x_{i+1/2})^2 f^{(3)}(\xi)\, d\xi \tag{5}$$

and $C_2 = 1/6$. Q.E.D.

In view of (4), it remains to find good bounds for the quantities $|e_i^{(1)} - e_{i+1}^{(1)}|$ ($i \in Z_N^e$), and we now consider this problem.

3.2. *Uniform Convergence*

THEOREM 4. *Let* $f \in C^1[a,b]$ *and* $f^{(1)} \in BV[a,b]$. (a) *Then* $|e_i^{(1)} - e_{i+1}^{(1)}| \leqslant 2\,\mathrm{Var}(f^{(1)})$ *for all* $i \in Z_N^e$. (b) *Then there exist constants* C_l, *independent of the partition, such that*

$$\|e^{(l)}\|_\infty \leqslant C_l \|\Delta\|^{1-l} [\|f^{(1)}\|_\infty + \mathrm{Var}(f^{(1)})] \tag{6}$$

for $l = 0$ *and* 1.

Proof. If $f \in C^1[a, b]$, we deduce from (2)

$$e_1^{(1)} - e_0^{(1)} = [f_1^{(1)} - f_0^{(1)}] + 2 \sum_{j=1}^{N-1} (-1)^j \frac{f_{j+1} - f_j}{h_j}. \tag{7}$$

Similar expressions can be obtained for $e_{i+1}^{(1)} - e_i^{(1)}$ for all $i \in Z_N^e$, and for simplicity we consider only $i = 0$. But $f_{j+1} - f_j = h_j f^{(1)}(\tau_j)$, where $\tau_j \in (x_j, x_{j+1})$. Then (7) becomes

$$e_1^{(1)} - e_0^{(1)} = [f_1^{(1)} - f_0^{(1)}] + 2 \sum_{j \in Z_N^0} \{f^{(1)}(\tau_{j+1}) - f^{(1)}(\tau_j)\}$$

and the first part is proved. The second part follows from the first and Propositions 2 and 3. Q.E.D.

The last theorem indicates that the remainder function is uniformly bounded and $\|f - s\|_\infty \to 0$ as $\|\Delta\| \to 0$. The following example shows that we cannot improve (6) without any supplementary hypothesis.

EXAMPLE. Consider $f(x) = \sin \pi x$, $x \in [0, 1]$, and Δ a uniform partition of $[0, 1]$. The symmetry implies $s_0^{(1)} = 0 = s_N^{(1)}$. But $f^{(1)}(0) = \pi = -f^{(1)}(1)$, so $|e_0^{(1)}| = \pi = |e_N^{(1)}|$ and (6) cannot be imporved (see Table II note the effect on $\|e\|_\infty$).

The next example shows that the estimate (6) can fail if the hypothesis of Theorem 4 is not satisfied, furthermore, we can improve it with stronger hypothesis.

EXAMPLE. Consider $f(x) = (1 + x)^{0.1} - (1 - x)^{0.1}$, $x \in [-1 + \varepsilon, 1 - \varepsilon]$. When $\varepsilon = 0$, the hypothesis of Theorem 4 is not satisfied and we do not

TABLE II

$f(x) = \sin \pi x$, $x \in [0, 1]$

N	$\|\Delta\| = \frac{1}{N}$	$\|e\|_\infty^{*\,(a)}$	$\|e^{(1)}\|_\infty^{*\,(a)}$
17	0.05882	0.4634E-1	3.1594
35	0.03030	0.2382E-1	3.1463
65	0.01538	0.1209E-1	3.1428
129	0.00775	0.6089E-2	3.1419
257	0.00389	0.3056E-2	3.1417
513	0.00195	0.1531E-2	3.1416
1025	0.00098	0.7662E-3	3.1417

[a] $\|e^{(l)}\|_\infty^*$ are estimations of $\|e^{(l)}\|_\infty$ and are computed according to $\|e^{(l)}\|_\infty^* = \max\{|e^{(l)}(y_{ij})| \mid y_{ij} = x_i + j\,(h_i/10), j = 0,\ldots, 9, \text{ and } i \in Z_N\}$.

observe (6) (see Table III, $K=0$, $\varepsilon=0$). When $\varepsilon=0.1$, we have $f\in C^{\infty}[-0.9,0.9]$, $f^{(1)}(-0.9)=f^{(1)}(0.9)$ and we observe a great improvement of (6) (see Table III, $K=0$, $\varepsilon=0.1$).

3.3. *The Uniform Case*

In this section we consider only uniform partitions. Hence Theorem 4 can be extended in the following way.

THEOREM 5. *Let* $k=1$ *or* 2, $f\in AC^{k+1,\infty}[a,b]$, $f^{(k+1)}\in BV[a,b]$, *and* Δ *a uniform partition of* $[a,b]$. (a) *Then there exists a constant* C_k *such that*

$$|e_i^{(1)}-e_{i+1}^{(1)}|\leqslant|f_N^{(1)}-f_0^{(1)}|+C_k\|\Delta\|^k\operatorname{Var}(f^{(k+1)})$$

for all $i\in Z_N^e$ ($C_1=1/2$ *and* $C_2=1/6$). (b) *Moreover, if* $f\in AC_p^{k+1,\infty}[a,b]$, *then there exist constants* C_{kl}, *independent of the partition, such that*

$$\|e^{(l)}\|_{\infty}\leqslant C_{kl}\|\Delta\|^{k+1-l}[\|f^{(k+1)}\|_{\infty}+\operatorname{Var}(f^{(k+1)})] \tag{8}$$

for all $l=0,\ldots,k+1$.

Proof. When $k=1$ or 2 and $f\in AC^{k+1,\infty}[a,b]$, we always have

$$f_{j+1}-f_j=\frac{h_j}{2}[f_{j+1}^{(1)}+f_j^{(1)}]-\int_{x_j}^{x_{j+1}}(\xi-x_{j+1/2})f^{(2)}(\xi)\,d\xi,$$

so (7) becomes

$$e_1^{(1)}-e_0^{(1)}=[f_N^{(1)}-f_0^{(1)}]-2\sum_{j=1}^{N-1}\frac{(-1)^j}{h_j}\int_{x_j}^{x_{j+1}}(\xi-x_{j+1/2})f^{(2)}(\xi)\,d\xi. \tag{9}$$

For a uniform partition Δ, the changes of variables $\eta=2(\xi-x_{j+1/2})/h_j$ ($\xi\in[x_j,x_{j+1}]$, $j\in Z_N$) yield to

$$e_1^{(1)}-e_0^{(1)}=[f_N^{(1)}-f_0^{(1)}]-\frac{\|\Delta\|}{2}\int_{-1}^{1}\eta\sum_{j\in Z_N^0}\left[f^{(2)}\left(x_{j+3/2}+\eta\frac{\|\Delta\|}{2}\right)\right.$$
$$\left.-f^{(2)}\left(x_{j+1/2}+\eta\frac{\|\Delta\|}{2}\right)\right]d\eta.$$

The result follows for $k=1$. When $k=2$, through integration by parts, (9) becomes

$$e_1^{(1)}-e_0^{(1)}=[f_N^{(1)}-f_0^{(1)}]-\sum_{j=1}^{N-1}\frac{(-1)^j}{h_j}\int_{x_j}^{x_{j+1}}\left[\frac{h_j^2}{4}-(\xi-x_{j+1/2})^2\right]f^{(3)}(\xi)\,d\xi \tag{10}$$

and, as before,

$$e_1^{(1)} - e_0^{(1)} = [f_N^{(1)} - f_0^{(1)}] - \frac{\|\Delta\|^2}{8} \int_{-1}^{1} (1-\eta^2) \sum_{j \in Z_N^0} \left[f^{(3)}\left(x_{j+3/2} + \eta \frac{\|\Delta\|}{2}\right) - f^{(3)}\left(x_{j+1/2} + \eta \frac{\|\Delta\|}{2}\right) \right] d\eta \tag{11}$$

and the proof of part (a) is completed. Part (b) is a direct consequence of (a) and Propositions 2 and 3. Q.E.D.

The following examples show that the hypotheses of Theorem 5 are essential.

EXAMPLE. Consider $f(x) = (1+x)^{K+0.1} - (1-x)^{K+0.1}$, $x \in [-1+\varepsilon, 1-\varepsilon]$, and $K = 1$ or 2. If $\varepsilon > 0$, then $f \in C_p^\infty[-1+\varepsilon, 1-\varepsilon]$ and we observe (12) in which $k = 2$ (see Table III). If $\varepsilon = 0$ then $f \notin AC_p^{k+1,\infty}[-1, 1]$ and the estimate (8) fails for $k = K$, but (8) is valid for $k = K - 1$ since $f \in AC_p^{K,\infty}[-1, 1]$ and $f^{(K)} \in BV[-1, 1]$ (see Table III).

EXAMPLE. We will construct a function $f \in AC_p^{k+1,\infty}[0, 1]$ such that $f^{(k+1)} \notin BV[0, 1]$ and for which there exists a family of uniform partitions leading to $e_1^{(1)} - e_0^{(1)} \simeq O(\|\Delta\|^{k-1})$.

Consider $k = 2$ (we essentially have the same situation when $k = 1$). In fact, we construct simultaneously f and an increasing family $\{\Delta_n\}_{n=1}^\infty$ of uniform partitions. If $\{k_n\}_{n=1}^\infty$ is a strictly increasing sequence of positive integers where $k_1 = 0$, we define the partition $\Delta_n = \{i3^{-k_n} \mid i = 0, \ldots, 3^{k_n}\}$. For each $n = 1, 2, \ldots$, let us define $f^{(3)}(x)$ for all $x \in (\|\Delta_{n+1}\|, \|\Delta_n\|]$ as follows:

$$f^{(3)}(x) = \frac{(-1)^{j-1}}{n} \quad \text{if} \quad \begin{cases} x \in (j3^{-k_{n+1}}, (j+1)3^{-k_{n+1}}], \\ j = 1, \ldots, 3^{k_{n+1}-k_n}. \end{cases}$$

It remains to choose $k_n (n \geqslant 2)$.

Assume $k_1, \ldots, k_n$ fixed, hence $f^{(3)}$ is defined on the interval $(\|\Delta_n\|, 1]$. It is easy to show that $f^{(3)}$ is of bounded variation over $(\|\Delta_n\|, 1]$, we will note this variation $\text{Var}_n(f^{(3)})$. Now let us use (11) with the partition Δ_{n+1}. Then

$$e_1^{(1)} - e_0^{(1)} = \frac{\|\Delta_{n+1}\|^2}{8} \int_{-1}^{1} (1-\eta^2) \times \left[\sum_{j=1}^{J-1} (-1)^{j+1} g_j(\eta) + \sum_{j=J}^{\bar{J}-1} (-1)^{j+1} g_j(\eta) \right] d\eta, \tag{12}$$

where

$$g_j(\eta) = f^{(3)}\left(x_{j+1/2} + \eta \frac{\|\Delta_{n+1}\|}{2}\right), \; J = 3^{k_{n+1}-k_n} \text{ and } \bar{J} = 3^{k_{n+1}}.$$

TABLE III

$$f(x) = (1+x)^{K+0.1} - (1-x)^{K+0.1},\ x \in [-1+\varepsilon, 1-\varepsilon]$$

K	ε	N	$\|\Delta\| = \frac{1}{N}$	$\|e^{(0)}\|_\infty$	$\|e^{(1)}\|_\infty$	$\|e^{(2)}\|_\infty$
0	0.1	17	0.10588	0.8194E−3	0.7281E−1	
		33	0.05455	0.1572E−3	0.2680E−1	
		65	0.02769	0.2333E−4	0.8053E−2	
		129	0.01395	0.3065E−5	0.2158E−2	
		257	0.00700	0.3844E−6	0.5523E−3	
		513	0.00351	0.4777E−7	0.1392E−3	
		1025	0.00176	0.5941E−8	0.3490E−4	
	0.0	17	0.11765	0.49455		
		33	0.06061	0.46281		
		65	0.03077	0.43248		
		129	0.01550	0.40383		
		257	0.00778	0.37693		
		513	0.00390	0.35176		
		1025	0.00195	0.32823		
1	0.1	17		0.6044E−4	0.5386E−2	
		33		0.9902E−5	0.1734E−2	
		65		0.1379E−5	0.4856E−3	
		129		0.1770E−6	0.1267E−3	
		257		0.2215E−7	0.3218E−4	
		513		0.2760E−8	0.8093E−5	
		1025		0.3444E−9	0.2029E−5	
	0.	17		0.1211E−2	0.7217	
		33		0.5837E−3	0.6753	
		65		0.2769E−3	0.6311	
		129		0.1303E−3	0.5893	
		257		0.6104E−4	0.5500	
		513		0.2854E−4	0.5133	
		1025		0.1333E−4	0.4790	
2	0.1	17		0.1787E−4	0.1648E−2	0.8928E−1
		33		0.2583E−5	0.4687E−3	0.4951E−1
		65		0.3414E−6	0.1241E−3	0.2615E−1
		129		0.4343E−7	0.3178E−4	0.1344E−1
		257		0.5455E−8	0.8025E−5	0.6812E−2
		513		0.6826E−9	0.2015E−5	0.3430E−2
		1025		0.8482E−10	0.5023E−6	0.1718E−2
	0.	17		0.8300E−4	0.799E−2	1.7556
		33		0.2044E−4	0.3824E−2	1.6398
		65		0.4898E−5	0.1806E−2	1.5307
		129		0.1158E−5	0.8477E−3	1.4285
		257		0.2719E−6	0.3966E−3	1.3329
		1025		0.1487E−7	0.8650E−4	1.1604

From the definitions of $f^{(3)}$ and $\mathrm{Var}_n(f^{(3)})$, (13) becomes

$$e_1^{(1)} - e_0^{(1)} \geqslant \frac{\|\Delta_{n+1}\|^2}{6} \left[\frac{1}{n}\left(\frac{\|\Delta_n\|}{\|\Delta_{n+1}\|} - 1\right) - \mathrm{Var}_n(f^{(3)})\right].$$

So if $k_{n+1}$is large enough, we deduce

$$e_1^{(1)} - e_0^{(1)} \geqslant \|\Delta_{n+1}\|^{1+(2/n)}.$$

Since we can do that for all $n = 1, 2, \ldots$, we define $f^{(3)} \in L^{\infty}[0, 1]$. It is easy to show that $f^{(3)} \notin BV[0, 1]$, and if we integrate $f^{(3)}$ and add some appropriate constants of integration, we obtain our desired function $f \in AC_p^{3,\infty}[0, 1]$.

3.4. *The Regular Case*

When the partition is not uniform, we generally cannot establish (8) without a stronger hypothesis. However, without any assumption on the partition Δ we can deduce from Proposition 3 this local result.

THEOREM 6. *Let $k = 1$ or 2 and $f \in AC_p^{k+1,\infty}[a, b]$. Then there exists at least one index i that possibly depends on the partition Δ and the function f, such that*

$$\max\{|e_i^{(1)}|, |e_{i+1}^{(1)}|\} \leqslant C_k \|f^{(k+1)}\|_\infty h_i^k,$$

$$\min\{|e_i^{(1)}|, |e_{i+1}^{(1)}|\} \leqslant \frac{C_k}{2} \|f^{(k+1)}\|_\infty h_i^k,$$

where $C_k = 1/(k+1)$. Moreover, there exist constants C_{kl} independent of the partition Δ, such that for almost all $x \in [x_i, x_{i+1}]$

$$|e^{(l)}(x)| \leqslant C_{kl} h_i^{k+1-l}$$

for all $l = 0, \ldots, k+1$.

Proof. Consider $Z_N = Z_N^+ \cup Z_N^-$, where $Z_N^+ = \{i \in Z_N \mid e_i^{(1)} \geqslant 0\}$ and $Z_N^- = \{i \in Z_N \mid e_i^{(1)} < 0\}$. Since N is odd, there exist at least two successive indices, with respect to Z_N, in Z_N^+ or in Z_N^-. Then we deduce the first two inequalities from (4) and the periodicity of $e^{(1)}$. These inequalities and Proposition 2 complete the proof. Q.E.D.

There exists a large class of functions for which (8), with $k = 1$, remains valid even for non-uniform partitions.

THEOREM 7. *Let* $f \in AC_p^{3,1}[a, b]$. (a) *Then* $\max\{|e_i^{(1)}|, |e_{i+1}^{(1)}|\} \leqslant (\|\Delta\|/2)\,\|f^{(3)}\|_1$ *for all* $i \in Z_N$. (b) *There exist constants* $C_l(\gamma)$, *that depend on the mesh ratio* γ, *such that*

$$\|e^{(l)}\|_\infty \leqslant C_l(\gamma)\,\|f^{(3)}\|_1\,\|\Delta\|^{2-l}$$

for all $l = 0, 1$ *or* 2.

Proof. Equations (5) and (10), respectively, to $|e_i^{(1)} + e_{i+1}^{(1)}| \leqslant (h_i/2)\,\|f^{(3)}\|_1$ and $|e_i^{(1)} - e_{i+1}^{(1)}| \leqslant (\|\Delta\|/4)\,\|f^{(3)}\|_1$ for all $i \in Z_N^e$. Hence (a) follows. To prove the second part, consider

$$e^{(2)}(x) = \frac{e_i^{(1)}}{h_i} - \frac{e_{i+1}^{(1)}}{h_i} - \frac{1}{h_i}\int_x^{x_{i+1}}\int_x^{\xi} f^{(3)}(\tau)\,d\tau\,d\xi.$$

Then $|e^{(2)}(x)| \leqslant (\gamma + 1)\,\|f^{(3)}\|_1$. Since there exists $\xi \in (x_i, x_{i+1})$ such that $e^{(1)}(\xi) = 0$, we have $e^{(1)}(x) = \int_\xi^x e^{(2)}(\tau)\, d\tau$ and $|e^{(1)}(x)| \leqslant h_i(\gamma + 1)\,\|f^{(3)}\|_1$ for all $x \in [x_i, x_{i+1}]$. Finally, since $e_i = 0$ $(i = 0, \ldots, N)$, we obtain $|e(x)| \leqslant ((\gamma + 1)/2)\, h_i^2\, \|f^{(3)}\|_1$. Q.E.D.

On the other hand, for the estimate (8) in which $k = 2$ the situation is quite different. Indeed, for a given smooth function it is easy to construct a regular family of partitions for which (8) fails.

EXAMPLE. Consider $f(x) = x^3/3!$, $x \in [-1, 1]$. Thus $f \in C_p^\infty[-1, 1]$, $f^{(3)}(x) = 1$ and (10) becomes

$$e_0^{(1)} - e_1^{(1)} = \frac{1}{6}\sum_{j \in Z_N^0} (h_{j+1}^2 - h_j^2).$$

For an arbitrary but fixed β, $0 < \beta < 1$, let us define the $h_i (i \in Z_N)$ as

$$\begin{aligned} h_i &= \|\Delta\| && \text{if} \quad i \in Z_N^e, \\ &= \beta\,\|\Delta\| && \text{if} \quad i \in Z_N^0, \end{aligned}$$

so that $\|\Delta\|\,[1 + (N - 1)(1 + \beta)/2] = 2$. Then $e_0^{(1)} - e_1^{(1)} = (\|\Delta\|^2/6)(N - 1)(1 - \beta^2)/2$. But $\|\Delta\|\,(N - 1)/2 \to 2/(1 + \beta)$ as $N \to \infty$, ensuring that $e_0^{(1)} - e_1^{(1)} = O(\|\Delta\|)$. This, together with (4), shows that $e_0^{(1)} - e_1^{(1)}$ are only $O(\|\Delta\|)$. A numerical example appears in Table IV with $\beta = 0.2$.

The last result, deduced from the preceding example, shows that the class of functions for which the estimate (8), with $k = 2$, fails is rather large.

THEOREM 8. *Let* $f \in C_p^3[a, b]$, $f^{(3)} \in BV[a, b]$ *and* f *is not a polynomial of degree* $\leqslant 2$. *Then there exists a constant* C *such that for all* $\gamma > 1$ *we can*

TABLE IV

$f(x) = x^3/3!,\ x \in [-1, 1]$

N	$\|\Delta\|$	$\|e^{(0)}\|_\infty^*$	$\|e^{(1)}\|_\infty^*$	$\|e^{(2)}\|_\infty^*$
17	0.18868	0.1075E-2	0.2575E-1	1.0755
33	0.09901	0.3106E-3	0.1336E-1	1.1980
65	0.05076	0.8371E-4	0.6811E-2	1.2640
129	0.02571	0.2175E-4	0.3439E-2	1.2982
257	0.01294	0.5542E-5	0.1728E-2	1.3157
513	0.00649	0.1400E-5	0.8659E-3	1.3245
1025	0.00325	0.351E-6	0.4335E-3	1.3289

choose a non-uniform partition of arbitrarily small mesh size $\|\Delta\|$ and of mesh ratio γ for which

$$e_0^{(1)} - e_1^{(1)} \geqslant \left(1 - \frac{1}{\gamma}\right) C \|f^{(3)}\|_\infty \|\Delta\| - \frac{\|\Delta\|^2}{6} \operatorname{Var}(f^{(3)}).$$

Proof. From the hypothesis, we can find a non-empty interval $[c, d] \subset [a, b]$ such that for all $x \in [c, d]$, $f^{(3)}(x) \geqslant (\|f^{(3)}\|_\infty/2)$. Let us take N odd and $\|\Delta\| = (b-a)/[1 + (1+\beta)(N-1)/2]$, where $0 < \beta = 1/\gamma < 1$. The knots of the partition Δ are then chosen as follows:

$$x_0 = a, \qquad x_1 = \|\Delta\|,$$

$$x_j = x_1 + \frac{j-1}{2}(1+\beta)\|\Delta\|, \qquad j \in Z_N^0,$$

$$\left.\begin{aligned} x_{j+1} &= x_j + \beta\|\Delta\| && \text{if } [x_j, x_{j+2}] \subset [c, d], \\ &= x_j + \frac{(1+\beta)}{2}\|\Delta\| && \text{if } [x_j, x_{j+2}] \not\subset [c, d], \end{aligned}\right\} \quad j \in Z_N^0.$$

If we note $\bar{Z}_N^0 = \{ j \in Z_N^0 \mid [x_j, x_{j+2}] \subset [c, d]\}$ and use the change of variables $\eta = 2(\xi - x_{j+1/2})/h_j$ ($\xi \in [x_j, x_{j+1}]$, $j \in Z_N$), then (10) becomes, for the partition Δ defined below,

$$e_0^{(1)} - e_1^{(1)} = \frac{\|\Delta\|^2}{8} \int_{-1}^{1} (1-\eta^2) \sum_{j \in \bar{Z}_N^0} \left(f^{(3)}\left(x_{j+3/2} + \frac{\eta}{2} h_{j+1}\right) - f^{(3)}\left(x_{j+1/2} + \frac{\eta}{2} h_j\right)\right) d\eta$$

$$+\frac{(1-\beta)\,\|\Delta\|}{8}\int_{-1}^{1}(1-\eta^2)\sum_{j\in\bar{Z}_N^0}'(h_j+h_{j+1})$$

$$\times f^{(3)}\left(x_{j+1/2}+\frac{\eta}{2}h_j\right)d\eta$$

$$+\frac{(1-\beta)^2\,\|\Delta\|^2}{32}\int_{-1}^{1}(1-\eta^2)\sum_{j\in Z_N^0\setminus\bar{Z}_N^0}'\left(f^{(3)}\left(x_{j+3/2}+\frac{\eta}{2}h_{j+1}\right)\right.$$

$$\left.-f^{(3)}\left(x_{j+1/2}+\frac{\eta}{2}h_j\right)\right)d\eta.$$

hence

$$e_0^{(1)}-e_1^{(1)}\geqslant\frac{(1-\beta)\,\|\Delta\|}{12}\,\|f^{(3)}\|_\infty\sum_{j\in\bar{Z}_N^0}'(h_j+h_{j+1})-\frac{\|\Delta\|^2}{6}\operatorname{Var}(f^{(3)}).$$

If $\|\Delta\|$ is small enough, we have $\sum_{j\in\bar{Z}_N^0}(h_j+h_{j+1})\geqslant(d-c)/2$ and the result follows with $C=(d-c)/24$. Q.E.D.

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